Magnetic String Coupled to Nonlinear Electromagnetic Field

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We introduce a class of rotating magnetically charged string solutions of the Einstein gravity with a nonlinear electrodynamics source in four dimensions. The present solutions has no curvature singularity and no horizons but has a conic singularity and yields a spacetime with a longitudinal magnetic field. Also, we investigate the effects of nonlinearity on the properties of the solutions and find that for the special range of the nonlinear parameter, the solutions are not asymptotic AdS. We show that when the rotation parameter is non zero, the spinning string has a net electric charge that is proportional to the magnitude of the rotation parameter. Finally, we use the counterterm method inspired by AdS/CFT correspondence and calculate the conserved quantities of the solutions.

I. INTRODUCTION

Topological defects are inevitably formed during phase transitions in the early universe, and their subsequent evolution and observational signatures must therefore be understood. The string model of structure formation may help to resolve one of cosmological mystery, the origin of cosmic magnetic fields [1]. There is strong evidence from all numerical simulations for the scaling behavior of the long string network during the radiation-dominated era. Apart from their possible astrophysical roles, topological defects are fascinating objects in their own right. Their properties, which are very different from those of more familiar system, can give rise to a rich variety of unusual mathematical and physical phenomena [2].

On another front, nonlinear electromagnetic fields are subjects of interest for a long time. For example, there has been a renewed interest in Born-Infeld gravity ever since new solutions have been found in the low energy limit of string theory. Static and rotating solutions of Born-Infeld gravity have been considered in Refs. [3–5].

In this paper, we turn to the investigation of spacetimes generated by static and spinning string sources in four-dimensional Einstein theory in the presence of a nonlinear electromagnetic field which are horinzonless and have nontrivial external solutions. The basic motivation for studying

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these kinds of solutions is that they may be interpreted as cosmic strings. Cosmic strings are topological structure that arise from the possible phase transitions to which the universe might have been subjected to and may play an important role in the formation of primordial structures. A short review of papers treating this subject follows. Solutions of Einstein's equations with conical singularities describing straight strings can easily be constructed [6]. One needs only a spacetime with a symmetry axis. If one then cuts out a wedge then a space with a string lying along the axis is obtained. A non axisymmetric solutions of the combined Einstein and Maxwell equations with a string has been found by Linet [7]. The four-dimensional horizonless solutions of Einstein gravity have been explored in [8, 9]. These horizonless solutions ([8, 9]) have a conical geometry; they are everywhere flat except at the location of the line source. The spacetime can be obtained from the flat spacetime by cutting out a wedge and identifying its edges. The wedge has an opening angle which turns to be proportional to the source mass. The extension to include the Maxwell field has also been done [10]. Static and spinning magnetic sources in three and four-dimensional Einstein-Maxwell gravity with negative cosmological constant have been explored in [11, 12]. The generalization of these asymptotically AdS magnetic rotating solutions to higher dimensions has also been done [13]. In the context of electromagnetic cosmic string, it has been shown that there are cosmic strings, known as superconducting cosmic strings, that behave as superconductors and have interesting interactions with astrophysical magnetic fields [14]. The properties of these superconducting cosmic strings have been investigated in [15]. Solutions with longitudinal and angular magnetic field were considered in Refs. [16–19]. Similar static solutions in the context of cosmic string theory were found in Ref. [20]. All of these solutions [16–18, 20, 21] are horizonless and have a conical geometry; they are everywhere flat except at the location of the line source. The extension to include the electromagnetic field has also been done [22, 23]. The generalization of these solutions in Einstein gravity in the presence of a dilaton and Born-Infeld electromagnetic fields has been done in Ref. [24].

Another example of the nonlinear electromagnetic field is conformally invariant Maxwell field. In many papers, straightforward generalization of the Maxwell field to higher dimensions one essential property of the electromagnetic field is lost, namely, conformal invariance. Indeed, in the Reissner-Nordström solution, the source is given by the Maxwell action which enjoys the conformal invariance in four dimensions. Massless spin-1/2 fields have vanishing classical stress tensor trace in any dimension, while scalars can be "improved" to achieve $T^{\alpha}_{\alpha} = 0$, thereby guaranteeing invariance under the special conformal (or full Weyl) group, in accord with their scale-independence [25]. Maxwell theory can be studied in a gauge which is invariant under conformal rescalings of the

metric, and at first, have been proposed by Eastwood and Singer [26]. Also, Poplawski [27] have been showed the equivalence between the Ferraris–Kijowski and Maxwell Lagrangian results from the invariance of the latter under conformal transformations of the metric tensor. Quantized Maxwell theory in a conformally invariant gauge have been investigated by Esposito [28]. In recent years, gravity in the presence of nonlinear and conformally invariant Maxwell source have been studied in many papers [29, 30].

The outline of our paper is as follows. We give a brief review of the field equations of Einstein gravity in the presence of cosmological constant and nonlinear electromagnetic field in Sec. II. In Sec. III we present static horizonless solutions which produce longitudinal magnetic field, compare it with the solutions of the standard electromagnetic field and then investigate the properties of the solutions and the effects of nonlinearity of the electromagnetic field on the deficit angle of the spacetime. Section IV will be devoted to the generalization of these solutions to the case of rotating solutions and use of the counterterm method to compute the conserved quantities of them. We finish our paper with some concluding remarks.

II. BASIC FIELD EQUATIONS

Our starting point is the four-dimensional Einstein-nonlinear Maxwell action

$$I_G = -\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}\,(R-2\Lambda-\alpha F^s) - \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\Theta(\gamma), \tag{1}$$

where R is the scalar curvature, Λ is the cosmological constant, F is the Maxwell invariant which is equal to $F_{\mu\nu}F^{\mu\nu}$(where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field and A_μ is the vector potential), α and s is a coupling and arbitrary constant respectively. The last term in Eq. (1) is the Gibbons-Hawking surface term. It is required for the variational principle to be well-defined. The factor Θ represents the trace of the extrinsic curvature for the boundary $\partial\mathcal{M}$ and γ is the induced metric on the boundary. Varying the action (1) with respect to the gravitational field $g_{\mu\nu}$ and the gauge field A_μ, yields

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \tag{2}$$

$$\partial_\mu\left(\sqrt{-g}F^{\mu\nu}F^{s-1}\right) = 0. \tag{3}$$

In the presence of nonlinear electrodynamics field, the energy-momentum tensor of Eq. (2) is

$$T_{\mu\nu} = 2\alpha\left[sF_{\mu\rho}F^{\rho}_{\nu}F^{s-1} - \frac{1}{4}g_{\mu\nu}F^s\right]. \tag{4}$$

The conserved mass and angular momentum of the solutions of the above field equations can be calculated through the use of the substraction method of Brown and York [31]. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. A well-known method of dealing with this divergence for asymptotically AdS solutions of Einstein gravity is through the use of counterterm method inspired by AdS/CFT correspondence [32]. In this Letter, we deal with the spacetimes with zero curvature boundary, $R_{abcd}(\gamma) = 0$, and therefore the counterterm for the stress energy tensor should be proportional to γ^{ab}. We find the suitable counterterm which removes the divergences as

$$I_{ct} = -\frac{1}{4\pi}\int_{\partial\mathcal{M}} d^3x \frac{\sqrt{-\gamma}}{l}. \tag{5}$$

Having the total finite action $I = I_G + I_{\rm ct}$, one can use the quasilocal definition to construct a divergence free stress-energy tensor [31]. Thus the finite stress-energy tensor in four-dimensional Einstein-nonlinear Maxwell gravity with negative cosmological constant can be written as

$$T^{ab} = \frac{1}{8\pi}\left[\Theta^{ab} - \Theta\gamma^{ab} + \frac{2\gamma^{ab}}{l}\right]. \tag{6}$$

The first two terms in Eq. (6) are the variation of the action (1) with respect to γ_{ab}, and the last term is the variation of the boundary counterterm (5) with respect to γ_{ab}. To compute the conserved charges of the spacetime, one should choose a spacelike surface $\mathcal{B}$ in $\partial\mathcal{M}$ with metric σ_{ij}, and write the boundary metric in ADM (Arnowitt-Deser-Misner) form:

$$\gamma_{ab}dx^a dx^a = -N^2dt^2 + \sigma_{ij}\left(d\varphi^i + V^i dt\right)\left(d\varphi^j + V^j dt\right),$$

where the coordinates φ^i are the angular variables parameterizing the hypersurface of constant r around the origin, and N and V^i are the lapse and shift functions, respectively. When there is a Killing vector field ξ on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (6) can be written as

$$Q(\xi) = \int_{\mathcal{B}} d^2x\sqrt{\sigma}T_{ab}n^a\xi^b, \tag{7}$$

where σ is the determinant of the metric σ_{ij}, ξ and n^a are, respectively, the Killing vector field and the unit normal vector on the boundary $\mathcal{B}$. For boundaries with timelike ($\xi = \partial/\partial t$) and rotational ($\varsigma = \partial/\partial\phi$) Killing vector fields, one obtains the quasilocal mass and angular momentum

$$M = \int_{\mathcal{B}} d^2x\sqrt{\sigma}T_{ab}n^a\xi^b, \tag{8}$$

$$J = \int_{\mathcal{B}} d^2x\sqrt{\sigma}T_{ab}n^a\varsigma^b. \tag{9}$$

These quantities are, respectively, the conserved mass and angular momentum of the system enclosed by the boundary $\mathcal{B}$. Note that they will both depend on the location of the boundary $\mathcal{B}$ in the spacetime, although each is independent of the particular choice of foliation $\mathcal{B}$ within the surface $\partial\mathcal{M}$.

III. STATIC NONLINEAR MAGNETIC STRING

Here we want to obtain the four dimensional solutions of Eqs. (2)-(4) which produce a longitudinal magnetic fields along the z direction. We assume the following form for the metric [11]

$$ds^2 = -\frac{\rho^2}{l^2}dt^2 + \frac{d\rho^2}{f(\rho)} + l^2 f(\rho) d\varphi^2 + \frac{\rho^2}{l^2}dz^2. \tag{10}$$

The function $f(\rho)$ should be determined and l has the dimension of length which is related to the cosmological constant Λ by the relation $l^2 = -3/\Lambda$. The coordinate z has the dimension of length and ranges $-\infty < z < \infty$, while the angular coordinate ϕ is dimensionless as usual and ranges in $0 \leq \phi < 2\pi$. The motivation for this curious choice for the metric gauge [$g_{tt} \propto -\rho^2$ and $(g_{\rho\rho})^{-1} \propto g_{\phi\phi}$] instead of the usual Schwarzschild gauge [$(g_{\rho\rho})^{-1} \propto g_{tt}$ and $g_{\phi\phi} \propto \rho^2$] comes from the fact that we are looking for magnetic solutions. Taking the trace of the gravitational field equation (2), the scalar curvature is expressed in terms of the Maxwell invariant F and cosmological constant Λ as

$$R = 2\left[\Lambda - \alpha(s-1)F^s\right].$$

Before studying in details the field equations, we first specify the sign of the coupling constant α in term of the exponent s in order to ensure a physical interpretation of our future solutions. In fact, the sign of the coupling constant α in the action (1) can be chosen such that the energy density, i.e. the $T_{\widehat{t}\widehat{t}}$ component of the energy-momentum tensor in the orthonormal frame, is positive

$$T_{\widehat{t}\widehat{t}} = \frac{\alpha}{2}F^s > 0.$$

As a direct consequence, one can show that the Maxwell invariant $F = \frac{2}{l^2}(F_{\phi\rho})^2$ is positive and hence, the sign of the coupling constant α should be positive, which can be set to 1 without loss of generality. It is well-known that the electric field is associated with the time component, A_t, of the vector potential while the magnetic field is associated with the angular component A_ϕ. From the above facts, one can expect that a magnetic solutions can be written in a metric gauge in which the components g_{tt} and $g_{\phi\phi}$ interchange their roles relatively to those present in the Schwarzschild

gauge used to describe electric solutions. The Maxwell equation (3) can be integrated immediately to give

$$F_{\phi\rho} = \begin{cases} 0, & s = 0, \frac{1}{2} \\ \frac{-2ql^2}{\rho}, & s = \frac{3}{2} \\ \frac{2(2s-3)ql^2}{(2s-1)\rho^{2/(2s-1)}}, & \text{otherwise} \end{cases}, \tag{11}$$

where q, an integration constant where the electric charge of the string is related to this constant for spinning string. Inserting the Maxwell fields (11) and the metric (10) in the field equation (2), we can simplify these equations as

$$\rho f^{'}(\rho) + f(\rho) + \Lambda\rho^2 - H(\rho) = 0 \tag{12}$$

where

$$H(\rho) = \begin{cases} 0, & s = 0, \frac{1}{2} \\ \frac{16q^3l^3\sqrt{2}}{\rho}, & s = \frac{3}{2} \\ 2^s(2s-1)\rho^2\left[\frac{2(2s-3)ql}{(2s-1)\rho^{(s-2)/(2s-1)}}\right]^{2s}, & \text{otherwise} \end{cases} \tag{13}$$

where the "prime" denotes differentiation with respect to ρ. One can show that these equations have the following solutions

$$f(\rho) = -\frac{\Lambda\rho^2}{3} + \frac{2ml^3}{\rho} + \begin{cases} 0, & s = 0, \frac{1}{2} \\ \frac{16q^3l^3\sqrt{2}(1+\ln\rho)}{\rho}, & s = \frac{3}{2} \\ \frac{2^{3s}(2s-3)^{2s-1}(ql)^{2s}}{2(2s-1)^{2s-2}\rho^{2/(2s-1)}}, & \text{Otherwise} \end{cases}, \tag{14}$$

where m is the integration constant which is related to mass parameter. In the linear case ($s = 1$), the solutions reduce to the asymptotically AdS horizonless magnetic string solutions for $\Lambda = -3/l^2$ [12]. Here, we want to investigate the effects of the nonlinearity on the asymptotic behavior of the solutions. It is worthwhile to mention that for $0 < s < \frac{1}{2}$, the asymptotic dominant term of Eq. (14) is third term and the solutions of the Einstein-nonlinear Maxwell field equations are not asymptotically AdS, but for the cases $s < 0$ or $s > \frac{1}{2}$ (include of $s = \frac{3}{2}$), the asymptotic behavior of Einstein-nonlinear Maxwell field solutions are the same as linear AdS case. Equations (11)-(14) show that the magnetic field is zero for the cases $s = 0, \frac{1}{2}$, and the metric function (14) does not possess a charge term and it corresponds to uncharged asymptotically AdS one.

Now, we want to investigate the special case, such that the electromagnetic field equation be invariant under conformal transformation ($g_{\mu\nu} \longrightarrow \Omega^2 g_{\mu\nu}$ and $A_\mu \longrightarrow A_\mu$). Consider the

Lagrangian of the form $L(F)$, where $F = F_{\mu\nu}F^{\mu\nu}$. It is easy to show that for this form of Lagrangian in 4-dimensions, $T_{\mu}^{\mu} \propto \left[F\frac{dL}{dF} - L\right]$; so $T_{\mu}^{\mu} = 0$ implies $L(F) = Constant \times F$. It is worthwhile to mention that only for linear case $s = 1$, the electromagnetic field equation is invariant under conformal transformation.

Here, we want to study the general structure of the solutions. One can find that the Kretschmann scalar, $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$, is

$$R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} = \left(\frac{d^2 f(\rho)}{d\rho^2}\right)^2 + 4\left(\frac{1}{\rho}\frac{df(\rho)}{d\rho}\right)^2 + 4\left(\frac{f(\rho)}{\rho^2}\right)^2.$$

It is easy to show that the Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverges at $\rho = 0$ and therefore one might think that there is a curvature singularity located at $\rho = 0$. However, as will be seen below, the spacetime will never achieve $\rho = 0$. Now, we look for the existence of horizons and, in particular, we look for the possible presence of magnetically charged black hole solutions. The horizons, if any exist, are given by the zeros of the function $f(\rho) = (g_{\rho\rho})^{-1}$. Let us denote the largest positive root of $f(\rho) = 0$ by r_0. The function $f(\rho)$ is negative for $\rho < r_0$, and therefore one may think that the hypersurface of constant time and $\rho = r_0$ is the horizon. However, the above analysis is wrong. Indeed, we first notice that $g_{\rho\rho}$ and $g_{\phi\phi}$ are related by $f(\rho) = g_{\rho\rho}^{-1} = l^{-2}g_{\phi\phi}$, and therefore when $g_{\rho\rho}$ becomes negative (which occurs for $\rho < r_0$) so does $g_{\phi\phi}$. This leads to an apparent change of signature of the metric from $+2$ to -2. This indicates that we are using an incorrect extension. To get rid of this incorrect extension, we introduce the new radial coordinate r as

$$r^2 = \rho^2 - r_0^2 \Rightarrow d\rho^2 = \frac{r^2}{r^2 + r_0^2}dr^2. \tag{15}$$

With this coordinate change the metric (10) is written as

$$ds^2 = -\frac{r^2 + r_0^2}{l^2}dt^2 + l^2 f(r)d\phi^2 + \frac{r^2}{(r^2 + r_0^2)f(r)}dr^2 + \frac{r^2 + r_0^2}{l^2}dz^2, \tag{16}$$

where the coordinates r assumes the values $0 \leq r < \infty$, and $f(r)$, is now given as

$$f(r) = -\frac{\Lambda\left(r^2 + r_0^2\right)}{3} + \frac{2ml^3}{(r^2 + r_0^2)^{1/2}} + \begin{cases} 0, & s = 0, \frac{1}{2} \\ \frac{8q^3 l^3 \sqrt{2}\left[2 + \ln\left(r^2 + r_0^2\right)\right]}{\left(r^2 + r_0^2\right)^{1/2}}, & s = \frac{3}{2} \\ \frac{2^{3s}(2s-3)^{2s-1}(ql)^{2s}}{2(2s-1)^{2s-2}(r^2 + r_0^2)^{1/(2s-1)}}, & \text{Otherwise} \end{cases}, \tag{17}$$

The electromagnetic field equation in the new coordinate is

$$F_{\phi r}=\begin{cases} 0, & s=0,\frac{1}{2} \\ \frac{-2ql^2}{\left(r^2+r_0^2\right)^{1/2}}, & s=\frac{3}{2} \\ \frac{2(2s-3)ql^2}{(2s-1)(r^2+r_0^2)^{1/(2s-1)}}, & \text{otherwise} \end{cases} . \tag{18}$$

One can show that all curvature invariants (such as Kretschmann scalar, Ricci scalar, Ricci square, Weyl square and so on) are functions of f'', f'/r and f/r^2. Since these terms do not diverge in the range $0 \leq r < \infty$, one finds that all curvature invariants are finite. Therefore this spacetime has no curvature singularities and no horizons. It is worthwhile to mention that the magnetic solutions obtained here have distinct properties relative to the electric solutions obtained in [30]. One can expect magnetic solutions from electric solution by a double Wick rotation such as $t \longrightarrow i\phi$ and $\phi \longrightarrow it/l$, ($i=\sqrt{-1}$). Indeed, the electric solutions have black holes, while the magnetic do not. However, the spacetime (16) has a conic geometry and has a conical singularity at $r=0$, since:

$$\lim_{r\to 0}\frac{1}{r}\sqrt{\frac{g_{\phi\phi}}{g_{rr}}}\neq 1. \tag{19}$$

That is, as the radius r tends to zero, the limit of the ratio "*circumference/radius*" is not 2π and therefore the spacetime has a conical singularity at $r=0$. The canonical singularity can be removed if one identifies the coordinate ϕ with the period

$$\text{Period}_{\phi}=2\pi\left(\lim_{r\to 0}\frac{1}{r}\sqrt{\frac{g_{\phi\phi}}{g_{rr}}}\right)^{-1}=2\pi(1-4\mu), \tag{20}$$

where μ is given by

$$\mu \;=\; \frac{1}{4}\left(1-\frac{2}{lr_0\left(\Omega-2\Lambda\right)}\right), \tag{21}$$

$$\Omega \;=\; \begin{cases} 0, & s=0,\frac{1}{2} \\ \frac{2^{11/2}q^3l^3}{r_0^3}, & s=\frac{3}{2} \\ \frac{8^s(2s-1)q^{2s}l^{2s}}{\left(\frac{2s-1}{2s-3}\right)^{2s}r_0^{4s/(2s-1)}}, & \text{otherwise} \end{cases} \tag{22}$$

The above analysis shows that near the origin $r=0$, the metric (16) describes a spacetime which is locally flat but has a conical singularity at $r=0$ with a deficit angle $\delta\phi=8\pi\mu$. Since near the origin the metric (16) is identical to the spacetime generated by a cosmic string, by using the Vilenkin procedure, one can show that μ of Eq. (21) can be interpreted as the mass per unit length of the string [33].

Also, In order to investigate the effects of the nonlinearity of the magnetic field on deficit angle $\delta\phi$, we plot it versus the charge parameter q in three figures. Fig. 1 shows that for $0<s<\frac{1}{2}$,

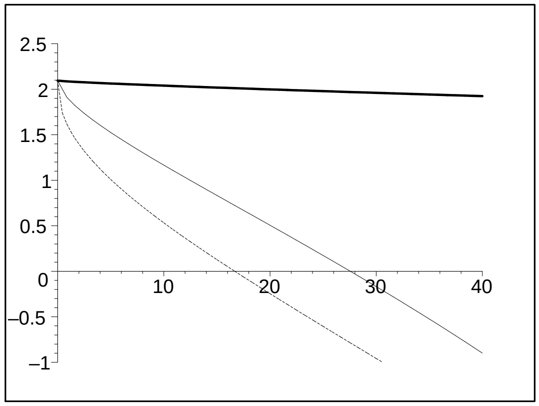

FIG. 1: The deficit angle versus q for $r_0 = 0.5$, $l = 1$, and $s = 0.2$ (dotted line), $s = 0.3$ (continuous line) and $s = 0.4$ (bold line).

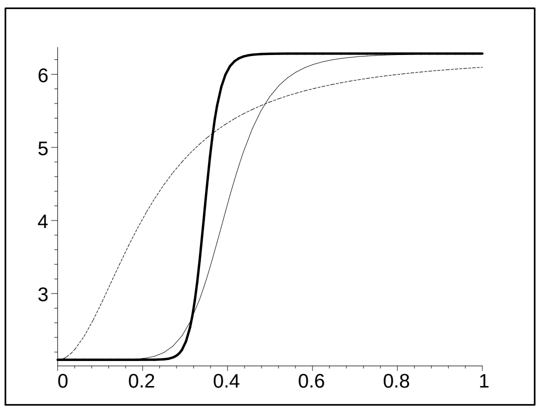

FIG. 2: The deficit angle versus q for $r_0 = 0.5$, $l = 1$, and $s = 1$ (dotted line), $s = 4$ (continuous line) and $s = 10$ (bold line).

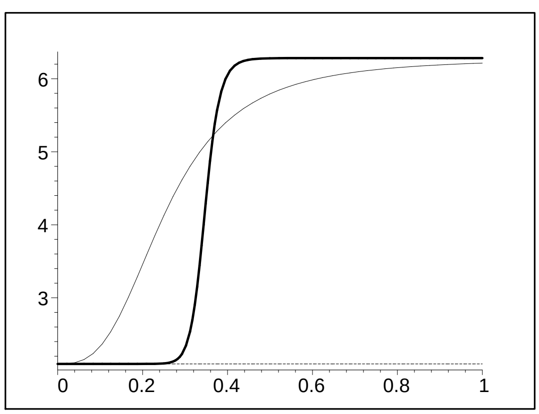

FIG. 3: The deficit angle versus q for $r_0 = 0.5$, $l = 1$, and $s = 0$ or $1/2$ (dotted line), $s = 3/2$ (continuous line) and $s = 10$ (bold line).

deficit angle decreases as the charge parameter of the spacetime, q, increases. But for a constant value of q, as the nonlinear parameter, s, increases, deficit angle increases too. Also, figures 2 and 3 show that for $s > \frac{1}{2}$, deficit angle increases as the charge parameter of the spacetime, q, increases and as the nonlinear parameter, s, increases, the rate of deficit angle growth increases too.

IV. SPINNING NONLINEAR MAGNETIC STRING

Now, we would like to endow the spacetime solutions (10) with a rotation. In order to add angular momentum to the spacetime, we perform the following rotation boost in the $t-\phi$ plane

$$t \mapsto \Xi t - a\phi, \qquad \phi \mapsto \Xi\phi - \frac{a}{l^2}t, \tag{23}$$

where a is a rotation parameter and $\Xi = \sqrt{1+a^2/l^2}$. Substituting Eq. (23) into Eq. (16) we obtain

$$\begin{aligned} ds^2 &= -\frac{r^2+r_0^2}{l^2}\left(\Xi dt - a d\phi\right)^2 + \frac{r^2 dr^2}{(r^2+r_0^2)f(r)} \\ &\quad + l^2 f(r)\left(\frac{a}{l^2}dt - \Xi d\phi\right)^2 + \frac{r^2+r_0^2}{l^2}dz^2, \end{aligned} \tag{24}$$

where $f(r)$ is given in Eqs. (17). The non-vanishing electromagnetic field components become

$$F_{rt} = -\frac{a}{\Xi l^2}F_{r\phi} = \begin{cases} 0, & s = 0, \frac{1}{2} \\ \frac{-2qa}{\Xi\left(r^2+r_0^2\right)^{1/2}}, & s = \frac{3}{2} \\ \frac{2(2s-3)qa}{\Xi(2s-1)(r^2+r_0^2)^{1/(2s-1)}}, & \text{otherwise} \end{cases}. \tag{25}$$

The transformation (23) generates a new metric, because it is not a permitted global coordinate transformation. This transformation can be done locally but not globally. Therefore, the metrics (16) and (24) can be locally mapped into each other but not globally, and so they are distinct. Note that this spacetime has no horizon and curvature singularity. However, it has a conical singularity at $r = 0$. It is notable to mention that for $s = 1$, these solutions reduce to asymptotically AdS magnetic rotating string solutions presented in [12].

The mass and angular momentum per unit length of the string when the boundary $\mathcal{B}$ goes to infinity can be calculated through the use of Eqs. (8) and (9). We find

$$M = \frac{\pi}{2}\left[3\Xi^2 - 2\right]m,$$

$$J = \frac{3\pi m \Xi l}{2}\sqrt{\Xi^2 - 1}.$$

For $a = 0$ ($\Xi = 1$), the angular momentum per unit length vanishes, and therefore a is the rotational parameter of the spacetime.

Finally, we compute the electric charge of the solutions. To determine the electric field one should consider the projections of the electromagnetic field tensor on special hypersurface. The electric charge per unit length Q can be found by calculating the flux of the electric field at infinity, yielding

$$Q = \sqrt{\Xi^2 - 1} \times \begin{cases} 0, & s = 0, \frac{1}{2} \\ \frac{3\sqrt{2}lq^2}{2\pi\Xi^2}, & s = \frac{3}{2} \\ \frac{-s\left(\frac{8(2s-3)lq}{(2s-1)\Xi}\right)^{2s-1}}{2^{3s+1}\pi l}, & \text{otherwise} \end{cases} . \tag{26}$$

It is worth noting that the electric charge is proportional to the rotation parameter, and is zero for the case of static solutions. This result is expected since now, besides the magnetic field along the ϕ coordinate, there is also a radial electric field ($F_{tr} \neq 0$).

V. CONCLUSIONS

In conclusion, with an appropriate combination of nonlinear electromagnetic field and Einstein gravity, we constructed a class of four dimensional magnetic string solutions which produces a longitudinal magnetic field. These solutions have no curvature singularity and no horizon, but have conic singularity at $r = 0$. In fact, we showed that near the origin $r = 0$, the metric describes a spacetime which is locally flat but has a conical singularity at $r = 0$ with a deficit angle $\delta\phi = 8\pi\mu$, where μ can be interpreted as the mass per unit length of the string. Also, we investigated the effects of nonlinearity on the deficit angle and asymptotic behavior of the solutions and found that for $0 < s < \frac{1}{2}$, the solutions are not asymptotically AdS and for $s < 0$ or $s > \frac{1}{2}$, the asymptotic behavior of solutions are the same as linear AdS case. In these static spacetimes, the electric field vanishes and therefore the string has no net electric charge. Then we added an angular momentum to the spacetime by performing a rotation boost in the $t - \phi$ plane. For the spinning string, when the rotation parameter is nonzero, the string has a net electric charge which is proportional to the magnitude of the rotation parameter. We also computed the conserved quantities of the solutions by using the conterterm method.

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